EXERCICE 7-4

I Tige

1) $\vec{OG} = \frac{L}{2}\vec{m_1} \Rightarrow \vec{V}(G) = \frac{d\vec{OG}}{dt} = \frac{d}{dt}\left(\frac{L}{2}\vec{m_1}\right) = \frac{L}{2}\frac{d\vec{m_1}}{dt} = \frac{L}{2}(\vec{\Omega}_{01} \wedge \vec{m_1})$

$= \frac{L}{2}(\dot\theta\vec{z} \wedge \vec{m_1}) = \frac{L}{2}\dot\theta\vec{y_1}$

2) tige (m, L) d'axe $\vec{m_1}$

$$\mathbb{I}(G, T, b_1) \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{mL^2}{12} & 0 \\ 0 & 0 & \frac{mL^2}{12} \end{pmatrix}$$

3) • $\vec{R}(V) = m\vec{V}(G) = m\frac{L}{2}\dot\theta\vec{y_1}$

• $\vec{\sigma}(G) = \mathbb{I}(G, T, b_1)\vec{\Omega}_T = \mathbb{I}(G,T,b_1)\vec{\Omega}_{01} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{mL^2}{12} & 0 \\ 0 & 0 & \frac{mL^2}{12} \end{pmatrix}_{b_1} \begin{pmatrix} 0 \\ 0 \\ \dot\theta \end{pmatrix} = {}_{b_1}\begin{pmatrix} 0 \\ 0 \\ \frac{mL^2}{12}\dot\theta \end{pmatrix}$

$\Rightarrow \vec{\sigma}(G) = \frac{mL^2}{12}\dot\theta\vec{z}$

4) • $\vec{R}(\gamma) = m\vec{\gamma}(G) = m\frac{d}{dt}(\vec{V}(G)) = m\frac{L}{2}\frac{d}{dt}(\dot\theta\vec{y_1})$

$\frac{d}{dt}(\dot\theta\vec{y_1}) = \ddot\theta\vec{y_1} + \dot\theta\frac{d\vec{y_1}}{dt} = \ddot\theta\vec{y_1} + \dot\theta(\vec{\Omega}_{01} \wedge \vec{y_1}) = \ddot\theta\vec{y_1} + \dot\theta(\dot\theta\vec{z} \wedge \vec{y_1})$

$= \ddot\theta\vec{y_1} - \dot\theta^2\vec{m_1}$

$$\vec{R}(\gamma) = m\frac{L}{2}(-\dot\theta^2\vec{m_1} + \ddot\theta\vec{y_1})$$

• $\vec{\delta}(G) = \frac{d\vec{\sigma}(G)}{dt} = \frac{mL^2}{12}\ddot\theta\vec{z}$

• $\vec{\delta}(C) = \vec{\delta}(G) + \vec{R}(\gamma) \wedge \vec{GC} = \frac{mL^2}{12}\ddot\theta\vec{z} + \frac{mL}{2}(-\dot\theta^2\vec{m_1} + \ddot\theta\vec{y_1}) \wedge \left(\frac{-L}{2}\vec{m_1}\right)$

$= \frac{mL^2}{12}\ddot\theta\vec{z} - \frac{mL^2}{4}\dot\theta^2\underbrace{(\vec{m_1} \wedge \vec{m_1})}_{\vec{0}} + \frac{mL^2}{4}\ddot\theta(\vec{y_1} \wedge \vec{m_1})$

$= \frac{mL^2}{12}\ddot\theta\vec{z} - \frac{mL^2}{4}\ddot\theta\vec{z} = \left(\frac{1}{12} - \frac{1}{4}\right)mL^2\ddot\theta\vec{z}$

$$\vec{\delta}(C) = -\frac{1}{6}mL^2\ddot\theta\vec{z}$$

5) $Ec = \frac{1}{2}m\vec{V}(G)^2 + \frac{1}{2}\vec{\Omega}_T \cdot \vec{\sigma}(G)$

$= \frac{1}{2}m\left(\frac{L}{2}\dot\theta\vec{y_1}\right)^2 + \frac{1}{2}\dot\theta\vec{z} \cdot \frac{mL^2}{12}\dot\theta\vec{z}$

$= \frac{1}{2}m\frac{L^2}{4}\dot\theta^2 + \frac{1}{2}\frac{mL^2}{12}\dot\theta^2 = \left(\frac{1}{8} + \frac{1}{24}\right)mL^2\dot\theta^2$

$$Ec = \frac{1}{6}mL^2\dot\theta^2$$

6) $E_p = mgz + cste$

z : altitude de $G \Rightarrow z = \vec{OG}\cdot(-\vec{x_0}) = -\frac{L}{2}\,\vec{x_1}\cdot\vec{x_0} = -\frac{L}{2}\cos\theta$

$E_p = -mg\frac{L}{2}\cos\theta + cste.$

7) on isole (T)

- Pesanteur $\quad \vec{P_T}\Big|_{b_0}\begin{matrix} mg \\ 0 \\ 0 \end{matrix} \qquad \vec{\mathcal{M}}_{P_T}(G) = \vec{0}$

- Liaison pivot en O
  Action en O $\quad \vec{R_0}\Big|\begin{matrix} X_0 \\ Y_0 \\ Z_0 \end{matrix} \qquad \vec{\mathcal{M}}_{R_0}(O)\Big|\begin{matrix} L_0 \\ M_0 \\ 0 \end{matrix}$

  on gardera b_0 $\leftarrow \begin{cases} b_0 \\ b_1 \\ b_2 \end{cases}$

- Liaison pivot en C
  Action de (D) sur (T) $\quad \vec{R_C}\Big|\begin{matrix} X_C \\ Y_C \\ Z_C \end{matrix} \qquad \vec{\mathcal{M}}_{R_C}(C)\Big|\begin{matrix} L_C \\ M_C \\ 0 \end{matrix}$

  idem $\leftarrow \begin{cases} b_0 \\ b_1 \\ b_2 \end{cases}$

PFD $\Rightarrow \begin{cases} \vec{R}_{F_e} = \vec{R}(T) \\ \vec{\mathcal{M}}_{F_e}(C) = \vec{\delta}(C) \end{cases}$ ↳ par exemple

$\vec{GC} = \frac{L}{2}\vec{x_1} = \frac{L}{2}(\cos\theta\,\vec{x_0} + \sin\theta\,\vec{y_0})$

- $\vec{\mathcal{M}}_{P_T}(C) = \vec{\mathcal{M}}_{P_T}(G) + \vec{P_T}\wedge\vec{GC} = \Big|_{b_0}\begin{matrix} 0 \\ 0 \\ 0 \end{matrix} + \Big|\begin{matrix} mg \\ 0 \\ 0 \end{matrix} \wedge \Big|\begin{matrix} \frac{L}{2}\cos\theta \\ \frac{L}{2}\sin\theta \\ 0 \end{matrix}$

$$\vec{\mathcal{M}}_{P_T}(C) = \Big|_{b_0}\begin{matrix} 0 \\ 0 \\ mg\frac{L}{2}\sin\theta \end{matrix}$$

- $\vec{\mathcal{M}}_{R_0}(C) = \vec{\mathcal{M}}_{R_0}(O) + \vec{R_0}\wedge\vec{OC} = \Big|_{b_0}\begin{matrix} L_0 \\ M_0 \\ 0 \end{matrix} + \Big|\begin{matrix} X_0 \\ Y_0 \\ Z_0 \end{matrix} \wedge \Big|\begin{matrix} L\cos\theta \\ L\sin\theta \\ 0 \end{matrix}$

$$\vec{\mathcal{M}}_{R_0}(C) = \Big|_{b_0}\begin{matrix} -LZ_0\sin\theta + L_0 \\ LZ_0\cos\theta + M_0 \\ L(X_0\sin\theta - Y_0\cos\theta) \end{matrix}$$

$\vec{y_1} = -\sin\theta\,\vec{x_0} + \cos\theta\,\vec{y_0}$

$$\Rightarrow \begin{cases} mg + X_0 + X_C = -m\frac{L}{2}\dot\theta^2\cos\theta - m\frac{L}{2}\ddot\theta\sin\theta \\ Y_0 + Y_C = -m\frac{L}{2}\dot\theta^2\sin\theta + m\frac{L}{2}\ddot\theta\cos\theta \\ Z_0 + Z_C = 0 \\ \\ L_0 - Z_0L\sin\theta + L_C = 0 \\ M_0 + Z_0L\cos\theta + M_C = 0 \\ mg\frac{L}{2}\sin\theta + L(X_0\sin\theta - Y_0\cos\theta) = -\frac{1}{6}mL^2\ddot\theta \end{cases}$$

II Disque

8) Disque $(9m, \frac{L}{3}) \Rightarrow MR^2 \Rightarrow 9m\frac{L^2}{9} = mL^2$

$$\mathbb{I}(C,D,b_2)\begin{pmatrix} \frac{mL^2}{4} & 0 & 0 \\ 0 & \frac{mL^2}{4} & 0 \\ 0 & 0 & \frac{mL^2}{2} \end{pmatrix}$$

9) • $\vec{R}(V) = 9m\,\vec{V}(C) = 9m\frac{d}{dt}(\vec{OC}) = 9m\frac{d}{dt}(L\vec{x_1}) = 9mL\dot{\theta}\vec{y_1}$

• $\vec{\sigma}(C) = \mathbb{I}(C,D,b_2)\,\vec{\Omega}_D = \mathbb{I}(C,D,b_2)\,\vec{\Omega}_{02}$

$$= \begin{pmatrix} \frac{mL^2}{4} & 0 & 0 \\ 0 & \frac{mL^2}{4} & 0 \\ 0 & 0 & \frac{mL^2}{2} \end{pmatrix}_{b_2}\begin{pmatrix} 0 \\ 0 \\ \dot{\theta}+\dot{\varphi} \end{pmatrix} = {}_{b_2}\begin{pmatrix} 0 \\ 0 \\ \frac{mL^2}{2}(\dot{\theta}+\dot{\varphi}) \end{pmatrix}.$$

$$\vec{\sigma}(C) = \frac{mL^2}{2}(\dot{\theta}+\dot{\varphi})\vec{z}$$

10) $Ec = \frac{1}{2}(9m)\,\vec{V}(C)^2 + \frac{1}{2}\vec{\Omega}_D \cdot \vec{\sigma}(C)$

$= \frac{1}{2}9m\,(L\dot{\theta}\vec{y_1})^2 + \frac{1}{2}(\dot{\theta}+\dot{\varphi})\vec{z} \cdot \frac{mL^2}{2}(\dot{\theta}+\dot{\varphi})\vec{z}$

$Ec = \frac{9}{2}mL^2\dot{\theta}^2 + \frac{1}{4}mL^2(\dot{\theta}+\dot{\varphi})^2$

11) $Ep = (9m)\,g\,z + cste$

z : altitude de $C \Rightarrow z\ \vec{OC}\cdot\vec{x_0} = L\vec{x_1}\cdot\vec{x_0} = L\cos\theta$

$Ep = 9mgL\cos\theta + CSTE$

12) • $\vec{R}(\gamma) = 9m\,\vec{\gamma}(C) = 9m\frac{d\vec{V}(C)}{dt} = 9mL(-\dot{\theta}^2\vec{x_1} + \ddot{\theta}\vec{y_1})$

• $\vec{\delta}(C) = \frac{d\vec{\sigma}(C)}{dt} = \frac{mL^2}{2}(\ddot{\theta}+\ddot{\varphi})\vec{z}$

13) on isole (D)

• Pesanteur $\vec{P}_D \begin{vmatrix} 9mg \\ 0 \\ 0 \end{vmatrix}_{b_0}$ $\quad \vec{M}_{P_D}(C) = \vec{0}$

• Liaison pivot en C
Action de (T) sur (D) $\vec{R'_c} \begin{vmatrix} -X_c \\ -Y_c \\ -Z_c \end{vmatrix}_{b_0}$ $\quad \vec{M}_{R'_c}(C) = \begin{vmatrix} -L_c \\ -M_c \\ 0 \end{vmatrix}_{b_0}$

• Action contact en I $\vec{R_I} \begin{vmatrix} -N_I \\ T_I \\ 0 \end{vmatrix}_{b_1}$ $\quad \vec{M}_{R_I}(I) = \vec{0}$

PFD $\begin{cases} \vec{R}_{Fc} = \vec{R}(\gamma) \\ \vec{M}_{Fc}(c) = \vec{\delta}(c) \end{cases}$

- $\vec{M}_{RI}(c) = \vec{M}_{RI}(I) + \vec{R}_I \wedge \vec{IC} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}_{b_1} + \begin{vmatrix} -N_1 \\ T_1 \\ 0 \end{vmatrix} \wedge \begin{vmatrix} -\frac{L}{3} \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ T_I \frac{L}{3} \end{vmatrix}_{b_1}$

- $\vec{R}(\gamma) = 9mL(-\dot{\theta}^2 \vec{x_1} + \ddot{\theta}\vec{y_1})$
$= 9mL[-\dot{\theta}^2(\cos\theta\,\vec{x_0} + \sin\theta\,\vec{y_0}) + \ddot{\theta}(-\sin\theta\,\vec{x_0} + \cos\theta\,\vec{y_0})]$
$= 9mL[(-\dot{\theta}^2\cos\theta - \ddot{\theta}\sin\theta)\vec{x_0} + (-\dot{\theta}^2\sin\theta + \ddot{\theta}\cos\theta)\vec{y_0}]$

- $\vec{R}_I = -N_1\vec{x_1} + T_1\vec{y_1} = -N_1(\cos\theta\,\vec{x_0} + \sin\theta\,\vec{y_0}) + T_1(-\sin\theta\,\vec{x_0} + \cos\theta\,\vec{y_0})$
$= (-N_1\cos\theta - T_1\sin\theta)\vec{x_0} + (-N_1\sin\theta + T_1\cos\theta)\vec{y_0}$

$$\begin{cases} 9mg - X_c - N_1\cos\theta - T_1\sin\theta = 9mL(-\dot{\theta}^2\cos\theta - \ddot{\theta}\sin\theta) \\ -Y_c - N_1\sin\theta + T_1\cos\theta = 9mL(-\dot{\theta}^2\sin\theta + \ddot{\theta}\cos\theta) \\ -Z_c = 0 \\ -L_c = 0 \\ -M_c = 0 \\ T_I \frac{L}{3} = m\frac{L}{2}(\ddot{\theta} + \ddot{\varphi}) \end{cases}$$

14] $\vec{U}_{D/0} = \vec{V}(I \in D) - \underbrace{\vec{V}(I \in S)}_{} = \vec{V}(I \in D)$

$\vec{V}(I) = \vec{V}(C) + \vec{\Omega}_D \wedge \vec{CI} = L\dot{\theta}\vec{y_1} + (\dot{\theta} + \dot{\varphi})\vec{z} \wedge \frac{L}{3}\vec{x_1}$
$= L\dot{\theta}\vec{y_1} + \frac{L}{3}(\dot{\theta} + \dot{\varphi})\vec{y_1} = \frac{L}{3}(4\dot{\theta} + \dot{\varphi})\vec{y_1}$

CRSG : $4\dot{\theta} + \dot{\varphi} = 0$

⚠ question 4] on ne peut pas utiliser la formule
$\vec{\delta}(c) = \frac{d\vec{\sigma}(c)}{dt}$ car c n'est ni le centre de gravité de T
ni fixe.